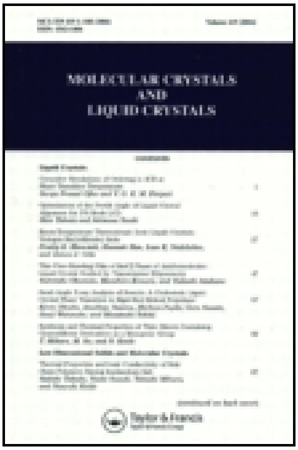

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On the Stability of Travelling Waves Arising in Nematic Liquid Crystals

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On the Stability of Travelling Waves Arising in Nematic Liquid Crystals

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The stability of travelling waves which occur when a nematic liquid crystal is subjected to crossed electric and magnetic fields has been studied previously where conditions on a control parameter q for stability to occur have been given. This article is concerned with the behaviour of the stable perturbations as time increases. For each of the three travelling wave solutions we can determine the long-term monotonic or oscillatory behaviour of the perturbations using knowledge of the spectrum of the operator governing the perturbation equation.

Keywords: travelling waves; perturbations; stabilty; spectra

INTRODUCTION

In this article we study an infinite sample of nematic liquid crystal which has a magnetic field $\mathbf{H}$ applied resulting in a static twist in the z-direction. An electric field $\mathbf{E}$ is then applied which causes the director to reorient via a travelling wave which, in turn, results in the motion of the twist in the z-direction. Recent theoretical work[1] has shown the existence of three types of travelling wave solutions to a cubic approximation of the usual nematic dynamic equations. The behaviour of perturbations to each of the aforementioned waves has been studied in detail[2]. Here we give a brief overview of the work and the main qualitative results. The problem to be considered is the time dependent solution to the dynamic equation for the director $\mathbf{n}$ in an infinite sample of nematic liquid crystal where the effects of bulk flow are ignored while the time dependent orientation of $\mathbf{n}$ is retained. The alignment of the director is assumed to be uniform in the xy-plane with the only change in orientation occurring with z. The crossed electric and magnetic fields are applied in the xy-plane. In particular we set

$$\mathbf{n} = (\cos\phi(z,t), \sin\phi(z,t), 0) \tag{1}$$

where ϕ is the angle $\mathbf{n}$ makes with the x-axis and introduce the $\mathbf{E}$ and $\mathbf{H}$ fields as

$$\mathbf{E} = E(\cos\beta, \sin\beta, 0), \qquad \mathbf{H} = H(1,0,0) \tag{2}$$

where E and H are the magnitudes of the fields and β is the angle between them with $0 \leq \beta \leq \pi/2$. It has been shown that the Leslie continuum equations for $\mathbf{n}$ reduce to

$$\gamma_1\phi_t = K_2\phi_{zz} - \frac{1}{2}\chi_a H^2\sin(2\phi) - \frac{1}{2}\epsilon_a\epsilon_0 E^2\sin 2(\phi - \beta) \tag{3}$$

where γ_1 is the twist viscosity coefficient and $K_2 > 0$ is the bulk elastic twist constant[1]. The diamagnetic anisotropy χ_a and the dielectric anisotropy ϵ_a are assumed to be positive while the (positive) permittivity of free space is denoted by ϵ_0. Equation (3) can be rescaled to give

$$\eta\phi_t = \xi^2\phi_{zz} - \frac{1}{2}\sin(2\phi - q) \tag{4}$$

where

$$\eta = \gamma_1(\epsilon_a^2\epsilon_0^2E^4 + \chi_a^2H^4 + 2\epsilon_a\epsilon_0\chi_aE^2H^2\cos(2\beta))^{-1/2} \tag{5}$$

$$\xi = \sqrt{K_2}(\epsilon_a^2\epsilon_0^2E^4 + \chi_a^2H^4 + 2\epsilon_a\epsilon_0\chi_aE^2H^2\cos(2\beta))^{-1/4} \tag{6}$$

$$q = \tan^{-1}\left(\frac{\epsilon_a\epsilon_0E^2\sin(2\beta)}{\chi_aH^2 + \epsilon_a\epsilon_0E^2\cos(2\beta)}\right). \tag{7}$$

The parameter q provides the key information on the relationship between the control parameters $\mathbf{E}$, $\mathbf{H}$ and β; q also characterises the solutions discussed below. We seek solutions for ϕ close to $\pi/2$ and so we set $\hat{\phi} = \phi - \pi/2$. This assumption is motivated by a previous argument[3]. Equation (4) can then be approximated by Taylor expanding the sine term up to cubic order in $\hat{\phi}$, making no restrictions on the control parameter q. This results in

$$\eta\hat{\phi}_t = \xi^2\hat{\phi}_{zz} - \frac{2}{3}(\cos q)F(\hat{\phi}, q) \tag{8}$$

where $F(\hat{\phi}, q)$ is a cubic in $\hat{\phi}$. Rescaling with $T = t\left(\frac{2\cos q}{3\eta}\right)$ and $Z = z\left(\frac{2\cos q}{3\xi^2}\right)^{1/2}$ then leads to

$$\hat{\phi}_T = \hat{\phi}_{ZZ} - F(\hat{\phi}, q). \tag{9}$$

Further detailed computations show that the cubic roots of F are real and are given by ϕ_1, ϕ_2 and ϕ_3 which are functions of the control parameter q and have been previously defined[1]. For our purposes it is necessary only to remark that we have the following inequalities whenever $0 \leq q < \pi/2$: $\phi_1 > 0$ and $\phi_3 < \phi_2 < \phi_1$. Hence we can write

$$F(\hat{\phi}, q) = (\hat{\phi} - \phi_1)(\hat{\phi} - \phi_2)(\hat{\phi} - \phi_3). \tag{10}$$

Travelling wave solutions are found by introducing the variable τ defined by $\tau = Z - cT + Z_0$ where c is specified for each solution and Z_0 is an arbitrary constant[1]. Equation (9) then becomes

$$\hat{\phi}_{\tau\tau} + c\hat{\phi}_\tau = F(\hat{\phi}, q). \tag{11}$$

Three types of travelling waves are known for (11) and we adopt the notation used elsewhere and label them as types A, B and C[1],[4]. Details of the travelling waves of types A and B can be found in [1] whilst, for brevity, we will consider only the type C travelling wave here.

Type C Travelling Waves

For $0 \leq q < \frac{\pi}{2}$ equation (11) has a type C travelling wave solution $\hat{\phi}$ which travels from ϕ_1 to ϕ_3 as $\tau \to \infty$ given by

$$\hat{\phi} = (\phi_1 - \phi_3)\left\{1 + \exp[\frac{1}{\sqrt{2}}(\phi_1 - \phi_3)\tau]\right\}^{-1} + \phi_3 \tag{12}$$

where $\phi_3 < \phi_1$ and $c = \frac{1}{\sqrt{2}}(\phi_1 + \phi_3 - 2\phi_2) > 0$. The type C travelling wave is shown in Fig. 1 for the value $q = \pi/4$. Travelling waves of types A and B behave in a similar manner and can be obtained by the substitutions $(\phi_2, \phi_3) \to (\phi_3, \phi_2)$ and $(\phi_2, \phi_3) \to (\phi_3, \phi_2)$ respectively.

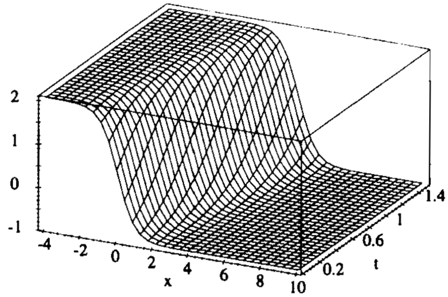

FIGURE 1 Type C travelling wave with $q = \pi/4$.

STABILITY

In order to investigate the stability of the type C travelling wave solutions we first, as is common, write (9) in a moving coordinate frame by changing the variables to $T = T$ and $\tau = Z - cT$. Equation (9) then becomes

$$\hat{\phi}_T = \hat{\phi}_{\tau\tau} + c\hat{\phi}_\tau - F(\hat{\phi}, q) \tag{13}$$

where now in general $\hat{\phi} = \hat{\phi}(\tau, T)$. When $\hat{\phi}(\tau)$ is a wavefront solution to (11) we consider solutions u to (13) of the form

$$u(\tau, T) = \hat{\phi}(\tau) + v(\tau, T) \tag{14}$$

where v is a small perturbation which depends on both τ and time T. Substituting (14) into (13) gives the linearised perturbation equation

$$v_T = v_{\tau\tau} + cv_\tau - \frac{\partial F}{\partial \hat{\phi}} v \tag{15}$$

where $\frac{\partial F}{\partial \hat{\phi}}$ is calculated directly from (10).
We examine perturbations which belong to $L_2(\mathbf{R})$ and allow our idea of stability to include not only small perturbations which die out, but also those perturbations which result in a small phase shift in the moving co-ordinate frame of the travelling wave. Following the procedure outlined in Grindrod[5], in this context $\hat{\phi}(\tau)$ is said to be stable if $u(\tau, T)$ as defined in (14) converges in $L_2(\mathbf{R})$ to some $\hat{\phi}(\tau + h)$ for some finite constant h as $T \to \infty$. If we define the operator A acting on perturbations $v \in L_2$ by

$$-Av = v_{\tau\tau} + cv_\tau - \frac{\partial F}{\partial \hat{\phi}} v \tag{16}$$

then $\hat{\phi}$ converges to a phase shifted travelling wave solution of the form $\hat{\phi}(\tau + h)$ whenever zero is a simple eigenvalue of A and the remainder of the spectrum lies in the complex half space $\{\lambda : \Re e(\lambda) \geq \delta\}$ for some $\delta > 0$. Further, an equilibrium solution u is said to be asymptotically stable if $\|v\|_{L_2} \to 0$ as $t \to \infty$. This property can be guaranteed when the spectrum of A, $\sigma(A)$, can be contained strictly to the right hand side of the complex plane, namely, $\Re e(\lambda) \geq \delta > 0$ for λ belonging to any part of the spectrum of A. In this case, it is generally known that $\|v\|_{L_2}$ decays like $e^{-\delta t}$ as $t \to \infty$ [5,p27]. Since the spectrum of A for the problems we shall consider consists of the essential spectrum, $\sigma_e(A)$, and the point spectrum (isolated eigenvalues of finite multiplicity), $\sigma_p(A)$, it follows that for stability we require that, for the class of perturbations considered, both the essential spectrum and the point spectrum of A in (16) lie to the right of the imaginary axis. We will consider the stability of the type C travelling waves and give comment on the stability of the remaining two travelling wave solutions.

Stability of Type C Travelling Waves

We construct an operator whose spectrum is contained wholly in the right-hand side of the complex plane. We can then study the corresponding eigenfunctions by transforming the perturbation equation into the form of the hypergeometric differential equation. This technique and similar methods have been used by Schlögl, Escher and Berry[6]. We analyse the solutions of this equation to obtain information about both the essential and pointwise spectra and therefore derive the stability properties of the perturbations.

Essential Spectrum

To gain information about the essential spectrum of an operator, reference [5] states

Theorem

Let B be the operator defined by $-Bv = Dv_{\tau\tau} - Mv_\tau - Nv$ where D is a positive constant, $M(\tau)$ and $N(\tau)$ are real bounded continuous functions and $M(\tau) \to M_\pm$, $N(\tau) \to N_\pm$ as $\tau \to \pm\infty$. Define

$$S_\pm = \{\lambda : Dk^2 + ikM_\pm + N_\pm - \lambda = 0; k \in \mathbf{R}\}. \tag{17}$$

Then $\sigma_e(B)$ lies in the region between and including S_+ and S_- in the complex plane. □

In this case, since $\phi_3 < \phi_2 < \phi_1$, as mentioned in the Introduction,

$$\frac{\partial F}{\partial \hat{\phi}} \to F_- \equiv (\phi_1 - \phi_2)(\phi_1 - \phi_3) > 0 \quad \text{as} \quad \tau \to \infty, \tag{18}$$

$$\frac{\partial F}{\partial \hat{\phi}} \to F_+ \equiv (\phi_3 - \phi_1)(\phi_3 - \phi_2) > 0 \quad \text{as} \quad \tau \to \infty. \tag{19}$$

Using the Theorem, we know that for type C travelling waves it follows that $\sigma_e(A)$ in (16) always lies to the right of the imaginary axis. Therefore we can use (17) to find the essential spectrum. By elimination of the parameter, k, between the real and imaginary parts of λ we can show that σ_e can be written as

$$\{\lambda : F_+ \leq \Re e(\lambda) - \left(\frac{\Im m(\lambda)}{c}\right)^2 \leq F_-\}. \tag{20}$$

We now consider the operator A which has the eigenvalue equation

$$v_{\tau\tau} + cv_\tau + \left(\lambda - \frac{\partial F}{\partial \hat{\phi}}\right) v = 0. \tag{21}$$

By transforming into the hypergeometric equation and applying the boundary conditions on v that $v(\tau) \to 0$ as $\tau \to \pm\infty$, we can find conditions for the point spectra of eigenvalues, both isolated and non-isolated, to exist.

Point spectrum consisting of isolated eigenvalues

There are two non-isolated eigenvalues. The first is given by

$$\lambda_L^C = -\frac{(\phi_1 - \phi_3)^2}{2}\left(r^2 + \frac{\sqrt{2}c}{(\phi_1 - \phi_3)}r\right) + F_-. \tag{22}$$

where r is given by

$$r = \frac{1}{2} + \frac{1}{2(\phi_1 - \phi_3)}(6\tan q - 5\phi_1 - 2\phi_2 - 5\phi_3). \tag{23}$$

The second eigenvalue is given by $\lambda_G^C = 0$.

Point spectrum consisting of non-isolated eigenvalues

Finally we obtain any non-isolated eigenvalues which may be present. We can satisfy the eigenvalue equation for any λ whenever

$$\Re e(\lambda) > \left(\frac{\Im m(\lambda)}{c}\right)^2 + F_+ = \lambda_1^C, \tag{24}$$

and

$$\Re e(\lambda) > \left(\frac{\Im m(\lambda)}{c}\right)^2 + F_- = \lambda_3^C. \tag{25}$$

Also, if $\lambda > \frac{c^2}{4} + F_+ = \lambda_2$ the damping of the perturbations will be oscillatory.

We now have enough knowledge of the complete possible spectrum of the operator A. As an example, we can examine the special case of $\Im m(\lambda) = 0$ to capture the typical qualities of the spectrum (the case where $\Im m(\lambda) \neq 0$ can be analysed similarly with care). Fig 2 shows the spectrum for real λ (i.e. $\Im m(\lambda) = 0$) as a function of q. For $\lambda_1^C \leq \lambda \leq \lambda_2^C$, shown by the dark grey region, we can have monotonic damping of the perturbations. However if $\lambda_2^C < \lambda \leq \lambda_3^C$, shown by the light grey region, the damping of the perturbations will be oscillatory. The dominant mode is the Goldstone mode at $\lambda_G^C = 0$ and, hence, for all $0 \leq q < \frac{\pi}{2}$ we have that the perturbations will induce a small phase shift to the original travelling wave. In some cases the resulting phase shifted solution may exhibit the superimposed oscillatory or monotonic decay of the original perturbation. As can be seen in Fig 4, λ_L^C corresponds to the next dominant mode until λ_L^C enters the continuous spectrum after which λ_1^C is preferred.

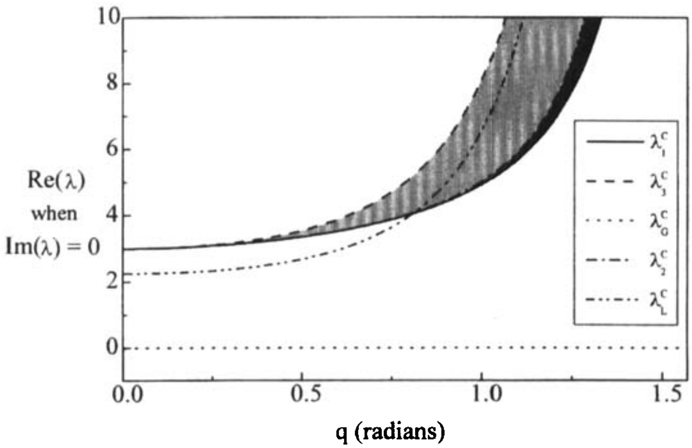

FIGURE 2 Spectrum for type C travelling wave.

Stability of Types A and B Travelling Waves

We can use a similar analysis to determine the stability of Types A and B travelling waves[2]. However, for both these waves part of the essential spectrum lies to the left of the imaginary axis. In an attempt to "move" the essential spectrum to the right of the imaginary axis we must restrict our choice of $v \in L_2$ by stipulating that v lies in a suitably weighted L_2 space. We require v to satisfy

$$\|v\|_{L_2^\mu} = \left\{ \int_R e^{2\mu\tau} \mid v(\tau, T) \mid^2 d\tau \right\}^{\frac{1}{2}} < \infty \tag{26}$$

where $\mu = \frac{c}{2}$ as has been previously determined[6].

The complete spectrum for perturbation equation to the type A travelling wave can be found such that the zero eigenvalue only exists whenever $q^* \leq q < \frac{\pi}{2}$ where $q^* \simeq 1.1$(radians) and the essential spectrum is always positive. There are no other eigenvalues present in the spectrum. Hence for $0 \leq q < q^*$ the perturbations in $L_2^\mu(\mathbf{R})$ will have decay whilst for $q^* \leq q < \frac{\pi}{2}$ the perturbations will induce a small phase shift to the original travelling wave.

For type B travelling waves the operator governing the perturbation equation consists only of the essential spectrum which is always positive. Therefore the perturbations in $L_2^\mu(\mathbf{R})$ will decay for all values $0 \leq q < \frac{\pi}{2}$ and so type B travelling waves are asymptotically stable.

CONCLUSIONS

The stability of each of the known travelling wave solutions can be ascertained using knowldege of the spectrum for the second order operator which governs the linearised perturbation equation. All the behaviour is dependent on the control parameter q which reflects the various possible combinations of **E**, **H** and the angle β between them. Physically, whenever the zero eigenvalue is not present the domain wall between two equilibrium states travels smoothly under small perturbations whilst whenever the zero eigenvalue is in the spectrum the domain wall will proceed in a different manner resulting in a smooth translation of the original wave.

There are several ways of adapting the work which has been presented here. A different approach to approximating the sine term in (4) has been studied[7] where the best fit cubic approximation through the original roots has been found. The subsequent analysis is similar to the work above and no significantly new stability results are obtained. Also solutions in a finite sample of smectic C liquid crystal have been studied numerically[8], whilst an analytic solution, which cannot be a travelling wave, has been found and the stability examined[9]. In such cases the spectrum of the linearised perturbation problem will not contain the essential spectrum since the problem is set in a finite domain.

References

[1] I.W. Stewart and T.R. Faulkner, *J. Phys. A: Math. Gen.* **28** 5643 (1995).
[2] G.J. Barclay and I.W. Stewart, *Cont. Mech. Thermodyn.* **10** 269 (1998).
[3] I.W. Stewart and T.R. Faulkner. *Cont. Mech. Thermodyn.* **9** 191 (1997).
[4] L. Lam, *Solitons in Liquid Crystals*, ed L. Lam and J. Prost (Springer-Verlag, New York, 1995) pp 9–50.
[5] P. Grindrod, *Patterns and Waves*, (Clarendon, Oxford, 1991).
[6] F. Schlögl, C. Escher and R. S. Berry, *Phys. Rev. A* **27** 2698 (1983).
[7] I.W. Stewart and T.R. Faulkner, to appear.
[8] J.E. Maclennan, N.A. Clark and M.A. Handschy, *Soliton in Liquid Crystals*, ed L. Lam and J. Prost (Springer-Verlag, New York, 1995) pp 151–190.
[9] D. Anderson and I.W. Stewart, to appear.